Proving that a function is increasing/decreasing on an interval.

Consider the function *f* defined as $f(x) = x^2 - 6x + 2$. Completing the square yields

$$f(x) = (x - 3)^2 - 7$$

from which we see that the minimum value of the function is -7 and that it occurs when $x = 3$. That is, the point $(3, -7)$ is the lowest point on the function's graph.

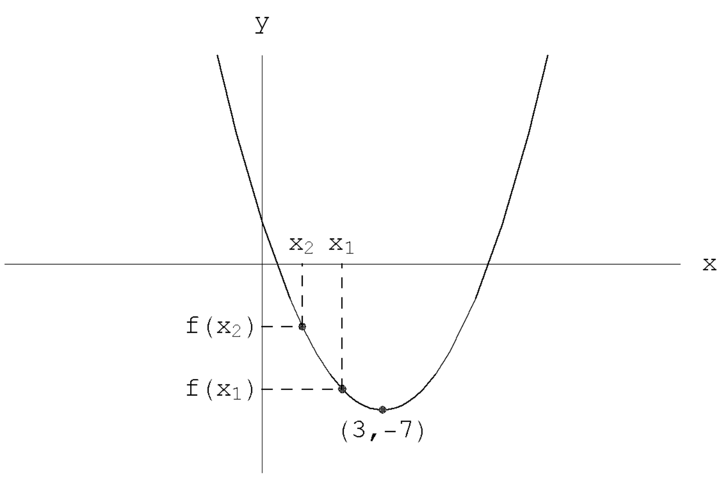
When we say that a function *f* is a decreasing function of *x* on an intverval (a, b), we mean that as *x* increases while remaining between *a* and *b*, *f(x)* decreases.

When we say that a function *f* is an increasing function of *x* on an intverval (a, b), we mean that as *x* increases while remaining between *a* and *b*, *f(x)* increases.

The graph of *f* suggests that in the interval $(-\infty, 3)$ as *x* increases, *f(x)* decreases; and that in the interval $(3, \infty +)$ as *x* increases, *f(x)* increases. Thus, *f* is decreasing on $(-\infty, 3)$ and *f* is increasing on $(3, \infty)$.

Now, suppose we wish to prove that *f* defined by $f(x) = (x-3)^2 - 7$ is decreasing on $(-\infty, 3)$. Our strategy is to show that for any x_1, x_2 where $x_2 < x_1 < 3$, $f(x_2) > f(x_1)$.

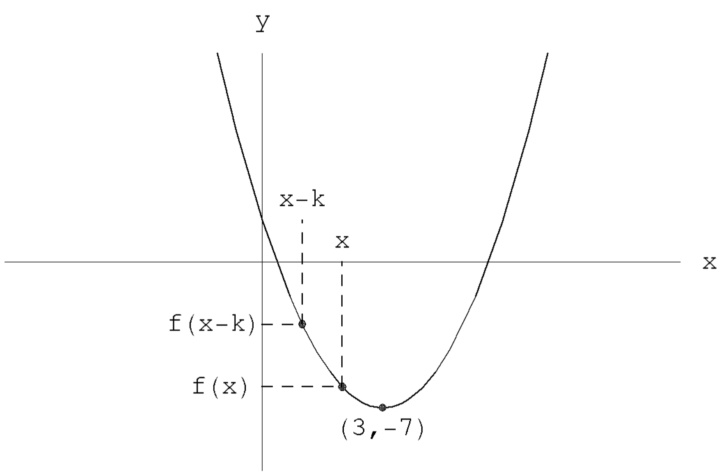
Here's the geometry



The function $f : f(x) = (x - 3)^2 - 7$ is decreasing on $(-\infty, 3)$.

Proof

Suppose $x < 3$ and $k > 0$. Then $x - k < x < 3$. This means $x - k$ is to the left of x and x is to the left of 3. If we can prove that $f(x - k) > f(x)$, then we will have shown that f is decreasing when $x < 3$. In other words, we will have shown that as long as we stay to the left of $x = 3$, the y-coordinate of a point moves down the y-axis as the x-coordinate of the point moves right on the x-axis



We must show that $f(x - k) > f(x)$

$$f(x - k) = (x - k - 3)^2 - 7 \quad (1)$$

$$= [(x - 3) - k]^2 - 7 \quad (2)$$

$$= (x - 3)^2 - 2(x - 3)k + k^2 - 7 \quad (3)$$

$$= -2(x - 3)k + k^2 + (x - 3)^2 - 7 \quad (4)$$

$$= [-2(x - 3)k + k^2] + f(x) \quad (5)$$

$$> f(x) \quad (6)$$

$\therefore \; f(x - k) > f(x)$

□

Here is why line 6 follows from line 5.

Since

$x < 3$,

$x - 3 < 3 - 3 = 0$.

That is,

$x - 3 < 0$.

Thus,

$(-2)(x - 3)\ k > 0$

and

$[-2\ (a - 3)\ k + k^2] > 0$.

Certainly, a number greater than zero plus f(x) is greater than f(x).